NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3768

ON POSSIBLE SIMILARITY SOLUTIONS FOR THREE-DIMENSIONAL INCOMPRESSIBLE LAMINAR BOUNDARY LAYERS

I - SIMILARITY WITH RESPECT TO STATIONARY RECTANGULAR COORDINATES

By Arthur G. Hansen and Howard Z. Herzig

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Cleveland, Ohio

Washington
October 1956

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SUMMARY

Solutions of mainstream flow patterns for all possible incompressible laminar-boundary-layer flows having classical similarity with respect to rectangular coordinate systems are derived. These solutions, which apply to a wide range of flows, are summarized in table form.

INTRODUCTION

Turbomachine boundary layers are quite generally turbulent except perhaps in local regions. As illustrated by the secondary-flow behavior described in references 1 to 3, these boundary-layer flows are three-dimensional as well. Mathematically, the nonlinear partial differential equations that describe this turbulent three-dimensional boundary-layer flow are all but intractable. Nevertheless, it is important, for turbomachine design procedures, to obtain theoretical solutions of the boundary-layer equations. In one much-used approach to this problem, by assuming two-dimensional laminar-boundary-layer flows, the equations are greatly simplified and solutions can be obtained. Actually, many useful applications of this two-dimensional laminar-boundary-layer flow theory have been made for estimating losses in turbomachines for design purposes. Often, however, this simplification is not physically acceptable. For example, when, in order to develop increased power output from compact engines, the flow velocities and mass-flow rates are increased, the boundary-layer secondary-flow effects likewise are increased. The boundary-layer accumulations and blade-end-region losses associated with the secondary flows soon constitute a substantial portion of the turbomachine losses. Unfortunately, these secondary-flow phenomena cannot be explained by extensions of the two-dimensional boundary-layer theory. In great measure, future progress in turbomachine design depends upon understanding and accounting for these secondary flows. As a consequence, the importance of understanding the three-dimensional turbulent-boundary-layer behavior has led to many experimental and theoretical investigations in recent years.

Although simplifying the boundary-layer equations by two-dimensionalizing them appears untenable for high-output engines, one most important result of the experimental investigations of references 1 to 3 is that laminar-boundary-layer behavior can usefully provide qualitative information concerning the turbulent-boundary-layer behavior. From a theoretical viewpoint, the importance of this observation can hardly be exaggerated. For the three-dimensional laminar-boundary-layer flows, the nonlinear partial differential equations are merely formidable; for the turbulent case the existence of turbulent fluctuation motions makes the determination of solutions much more complicated. The study of three-dimensional laminar-boundary-layer flows may, therefore, yield important information for the turbomachine designer. In addition, it is well to recognize that the external-flow wing-boundary-layer problem for high-altitude flight is also essentially a three-dimensional laminar-boundary-layer (secondary-flow) problem. (The existence of a limiting cross-channel-flow streamline depicted in references 1 and 2 eliminates consideration of a nonviscous-flow theory, at least in regions where it is reasonable to expect thin boundary layers.)

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Consequently, theoretical investigations of three-dimensional laminar-boundary-layer flows over a surface having a leading edge have been made (refs. 4 to 8, e.g.). Two types of solutions are obtained: approximate solutions (perturbation method) for general mainstream flows confined to regions of small turning (refs. 4 and 7) and exact similarity solutions (for boundary-layer flows having affine velocity profiles) for restricted types of mainstream flows but with no restriction on turning (refs. 5, 6, and 8). The most general of these similarity solutions (ref. 8) is applicable for quite arbitrary main-flow streamline paths, provided the axial velocity component is constant (that is, constant static pressure axially) and the main-flow streamlines are all translates (no variation of streamline shapes in the tangential direction).

The present investigation seeks to determine what further extensions of the similarity solutions are possible. New similarity parameters η are defined corresponding to mainstream flows not previously considered that enable the boundary-layer equations to be reduced to ordinary differential equations in terms of functions of these parameters. (Symbols are defined in appendix A.) These η's are found as a result of a systematic analysis using group theoretic approaches ("search for symmetric solutions" and "inverse" method, ref. 9) described in appendix B of this report. The evaluation of the functions of η and the determination of the corresponding three-dimensional boundary-layer flow paths remain as problems of numerical analysis which are not considered here. Thus, while the integrations of the equations are not carried out, the similarity solution may be considered essentially complete for present purposes upon reduction of the boundary-layer equations to ordinary differential equations.

Solutions obtained include cases of accelerating or decelerating main flows for quite general streamline paths, not confined to flow fields of streamline translates. These solutions are summarized in table I, which includes the mainstream velocity components, the generalized similarity parameter, and the final set of ordinary differential equations for each family of mainstream flows.

Similarity solutions obtained in previous investigations (refs. 4 to 8 and 10 to 15) are likewise cited in the table to show their relation to the solutions obtained in the present investigation.

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SIMILARITY SOLUTIONS FOR GENERALIZED SIMILARITY PARAMETER η

Similarity with Respect to Rectangular Coordinates

The equations describing the steady incompressible thin-laminar-boundary-layer flow over a flat or slightly curved surface with coordinate axis oriented as shown in this sketch

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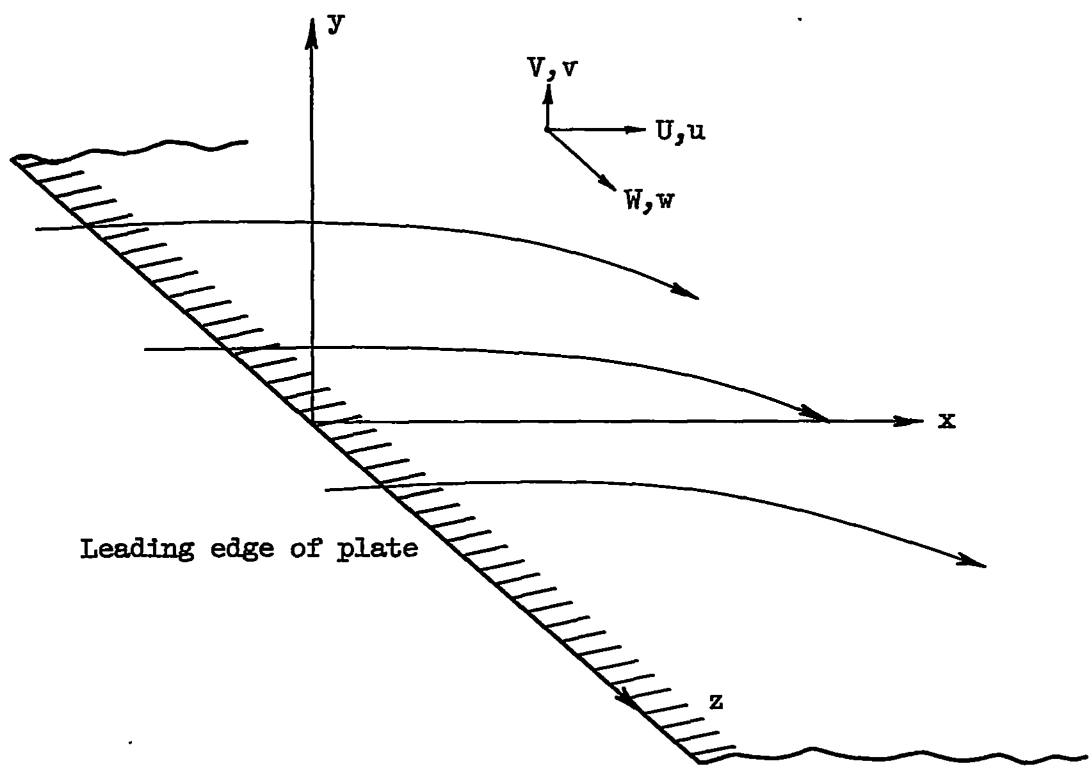

may be written

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - \nu \frac{\partial^2 u}{\partial y^2} = U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + W \frac{\partial U}{\partial z} \tag{1a}$$

$$u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} - \nu \frac{\partial^2 w}{\partial y^2} = U \frac{\partial W}{\partial x} + V \frac{\partial W}{\partial y} + W \frac{\partial W}{\partial z} \tag{1b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1c}$$

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The boundary conditions are

$$u = v = w = 0 \quad \text{for} \quad y = 0 \text{ (at surface)}$$

$$\lim_{y \to \infty} u = U \qquad \lim_{y \to \infty} w = W$$

It is known from the results of references 4 to 8 that solutions involving similarity parameters η have been obtained for the incompressible three-dimensional laminar-boundary-layer development on a nearly flat surface with a leading edge for certain restricted main flows. The purpose of the present investigation was to try to find solutions for much more general flows than had been obtained heretofore by use of more generalized similarity parameters η. As in appendix B and references 4 to 8, the dimensionless velocity components u/U and w/W are defined in terms of functions $F(\eta)$ and $G(\eta)$, respectively, where $\eta = \eta(x,y,z)$. The continuity equation (1c) may then be integrated, solving for v; and equations (1a) and (1b) may be transformed (by substitution) to the new coordinates, x, z, and η. The approach used herein is to determine the conditions under which the transformed equations (1a) and (1b) become ordinary differential equations in $F(\eta)$ and $G(\eta)$, which can then be integrated numerically. For the purposes of this report, the similarity solutions will be considered completed when the defining equations have been reduced to ordinary differential equations. The actual numerical integrations will not be performed here.

Generalized Similarity Parameter η

When the process outlined above is attempted, several practical restrictions appear at once. Unless the similarity parameter $\eta(x,y,z)$ has y separable, that is,

$$\eta = f(y)g(x,z)$$

the transformed equations are inordinately complicated. Furthermore, in order to be able to integrate the continuity equation (1c) readily, η must in addition be linear in y. Thus, η is chosen as

$$\eta \equiv \frac{y}{\sqrt{\nu}} g(x,z) \tag{2}$$

and the integration can then be carried through. This definition of η is not as restrictive as it first appears, because any similarity parameter $\eta^* \equiv \eta^*(x,y,z)$ can be used for which a functional relation can be found such that

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$$f(\eta^*) = \eta = \frac{y}{\sqrt{\nu}} g(x,z)$$

The functional relations of F and G to η^* are determined by

$$F = F(\eta) = F\left[f(\eta^*)\right]$$

$$G = G(\eta) = G\left[f(\eta^*)\right]$$

For an example, one could choose

$$\eta^* = y^r\sqrt{x^r z^{n-r}}$$

(suggested in appendix B by the similitude transformation). The transformation

$$(\eta^*)^{1/r} = \eta = y(x^r z^{n-r})^{1/2r}$$

determines the corresponding parameter η linear in y for use in the integrations.

Thus any solutions obtained in the present investigation are classical similarity solutions in the sense that, like all previous solutions involving a similarity parameter, the parameter is linear in y.

Functions $F(\eta)$ and $G(\eta)$ and Boundary Conditions

For flows over a nearly flat surface with thin boundary layers such as are being considered here, U and W are to be functions of x and z only, $U = U(x,z)$, $W = W(x,z)$.

As indicated earlier, it is assumed that the velocity components u and w are expressible as

$$u = U(x,z)\ F'(\eta) \tag{4a}$$

and

$$w = W(x,z)\ G'(\eta) \tag{4b}$$

The stipulation that u and w vanish on the bounding surface and approach the mainstream velocity at great distances from the boundary requires that

$$F'(0) = G'(0) = 0$$

$$\lim_{y\to\infty} F'(\eta) = 1$$

$$\lim_{y\to\infty} G'(\eta) = 1$$

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The form of the component v can now be determined from the continuity equation. Substitution of equations (4a) and (4b) into equation (1c) yields

$$\frac{\partial U}{\partial x} F' + UF'' \frac{\partial \eta}{\partial x} + \frac{\partial W}{\partial z} G' + WG'' \frac{\partial \eta}{\partial z} + \frac{\partial v}{\partial y} = 0 \tag{5}$$

Equation (5) can now be solved for v using equation (2), and the resulting solution is given by

$$\frac{v}{\sqrt{\nu}} = -\frac{1}{g}\left(\frac{\partial U}{\partial x} F + \frac{\partial W}{\partial z} G\right) + U \frac{\partial g^{-1}}{\partial x} (\eta F' - F) + W \frac{\partial g^{-1}}{\partial z} (\eta G' - G) + f(x,z) \tag{4c}$$

For v to vanish on the bounding surface, it can be required without loss of generality that $F(0) = G(0) = 0$ and that $f(x,z) \equiv 0$. (Appendix C contains a discussion of the necessary and sufficient boundary conditions on F and G.)

Ordinary Differential Equations

Substituting equations (2) and (4) into equations (1a) and (1b) followed by simplification then gives the following two differential equations in $F(\eta)$ and $G(\eta)$:

$$\frac{\partial U}{\partial x} (F'^2 - FF'' - 1) + W \frac{\partial \ln U}{\partial z} (G'F' - 1) - \frac{\partial W}{\partial z} (GF'') +$$

$$U \frac{\partial \ln g^2}{\partial x} \left(\frac{FF''}{2}\right) + W \frac{\partial \ln g^2}{\partial z} \left(\frac{GF''}{2}\right) - g^2 F''' = 0 \tag{6}$$

$$\frac{\partial W}{\partial z}\left(G'^2 - GG'' - 1\right) + U\,\frac{\partial \ln W}{\partial x}\left(F'G' - 1\right) - \frac{\partial U}{\partial x}\left(FG''\right) +$$

$$W\,\frac{\partial \ln g^2}{\partial z}\left(\frac{GG''}{2}\right) + U\,\frac{\partial \ln g^2}{\partial x}\left(\frac{FG''}{2}\right) - g^2G''' = 0 \qquad (7)$$

As previously stated, the final goal of the similarity transformations is to convert the partial differential equations (1) into ordinary differential equations in F, G, and their derivatives only. Whether or not equations (6) and (7) become ordinary differential equations, however, will ultimately depend upon the nature of U, W, and g^2, as these quantities appear in the coefficients of all terms involving F, G, and their derivatives. The necessary requirement for retaining only terms in F, G, and their derivatives in equations (6) and (7) is that the nonconstant coefficients of the various terms be proportional. If such proportionality exists, the coefficients can be divided out of the equations, and a system of ordinary differential equations in η will remain. The condition for obtaining ordinary differential equations would then be equivalent to the following system of partial differential equations for the functions U, W, and g^2:

$$\frac{\partial U}{\partial x} = a_1 W\,\frac{\partial \ln U}{\partial z} = a_2\,\frac{\partial W}{\partial z} = a_3 U\,\frac{\partial \ln g^2}{\partial x} = a_4 W\,\frac{\partial \ln g^2}{\partial z} = a_5 g^2 = a_6 U\,\frac{\partial \ln W}{\partial x} \qquad (8)$$

where $a_i \neq 0$ for $i = 1, \ldots, 6$. The solutions of equation (8) describe all possible mainstream flows of the type $U = U(x,z)$, $W = W(x,z)$ for which solutions of the corresponding three-dimensional boundary-layer equations could be obtained by the usual (i.e., linear in y) type of similarity transformation in rectangular coordinates.

In the following section, certain consequences of requiring proportionality between the coefficients in equations (6) and (7) are investigated. Whenever such proportionality between two coefficients is assumed, it is referred to as an "ordinary-differential-equation condition" and is abbreviated "o.d.e. condition." (The o.d.e. conditions for the incompressible two-dimensional boundary-layer flows are discussed in refs. 15 to 17. The o.d.e. conditions for the compressible two-dimensional boundary-layer flows are discussed in refs. 18 to 20 where systems of equations in two similarity parameters are obtained which closely parallel eqs. (6) and (7) of the present investigation.)

Solutions for U, W, and g^2 depend upon the o.d.e. conditions and the system of differential equations in these quantities which results. However, the o.d.e. conditions in turn may depend upon certain basic assumptions regarding the nature of U, W, and g^2. For example, if it is assumed initially that $U = U(x)$, the coefficient $W \frac{\partial \ln U}{\partial z}$ will not appear in an o.d.e. condition. In general, it will be necessary to make such initial assumptions about the form of at least one of the unknown functions before a unique set of o.d.e. conditions can be determined.

The calculation of all functions can be carried out in a straightforward manner by making an initial assumption on the form of U or W, by setting up the o.d.e. conditions, and by solving the corresponding differential equations. The cases where it is assumed that $U = U(x,z)$ or $W = W(x,z)$ are more complicated than cases resulting from other possible assumptions on the nature of U or W because a larger number of o.d.e. conditions result from equation (8) (if for no other reason).

As an example of how calculation of the functions is actually carried out, the most complicated case, the case of initially assuming $U = U(x,z)$, $W = W(x,z)$, and $g \neq 0$, is presented in appendix D.

RESULTS AND DISCUSSION

Mainstream Configurations

As a result of the analysis above, it can be shown that four families of solutions to equation (8) are obtained corresponding to

$$\left.\begin{aligned} U &= me^{nx}; & W &= ae^{rx} \\ U &= mx^{n}; & W &= px^{r} \\ U &= me^{nx}z^{r-1}; & W &= ae^{nx}z^{r} \\ U &= mx^{n}z^{r-1}; & W &= px^{n-1}z^{r} \end{aligned}\right\} \qquad (9)$$

(g^2 is proportional to $\partial U/\partial x$ in all cases). It is interesting to note the similarity between these results and the results obtained for the two-dimensional case in reference 15 (pp. 116-120). For this case, the forms for $U(x)$ (W is identically zero) are shown to be

$$U = cx^{m}$$

and

$$U = ce^{px}$$

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For convenience, the four basic solutions along with five other forms (which, it turns out, correspond to the four basic forms under rotation of the coordinate axes) are presented in table I. Table I, which includes within its organization all the possible solutions for mainstream flows having classical similarity with respect to rectangular coordinates, for $U = U(x,z)$ and $W = W(x,z)$, is discussed in the following paragraphs.

U and W. - U and W, the velocity components of the mainstream in the x and z directions, respectively, are listed first in the table.

For all except case VI (obtained from case IV by rotation), the chart indicates U and W are representable by (a) powers of linear functions in x or z individually, (b) exponential functions in x or z alone, or (c) the products of such functions. In case VI, U and W take the forms of powers of linear combinations of x and z resulting in mixed polynomials in x and z. It may be observed that great freedom of choice of U and W velocity components can be obtained by assigning different values to the available constants.

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Projection of mainstream on surface. - The equation for the projection of the mainstream on the surface may be obtained by integrating $dz/dx = W/U$. For flows nearly parallel to the surface, this projection approximates the actual main-flow streamline path. When the constants are chosen such that straight mainstream flows result (always true for cases III and VI), there can be no boundary-layer crossflow relative to the mainstream. For such flows, $F = G$ and equations (6) and (7) are identical. It may be observed that, except for solutions cited under case IV and the stagnation solution of reference 10, only solutions having straight streamlines (in the x-z plane) have been completed.

Irrotationality. - This listing indicates choices of constants for which the mainstream component of vorticity normal to the surface $(\partial U/\partial z - \partial W/\partial x)$ is zero. For the flows considered here, only this component of vorticity can be much different from zero, in any event. Therefore, specifying $\partial U/\partial z - \partial W/\partial x = 0$, as is done for the listing of irrotationality, actually serves to set the conditions for nearly irrotational main flow.

Irrotationality and two-dimensional continuity. - The flows represented by cases III and VI are straight flows. Obviously these flows can be irrotational and have two-dimensional continuity $\left(\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0\right)$ only for the case of constant velocity. The combined conditions would require straight constant-speed flows for cases I, II, IV, V, VII, and VIII as well. Only case IX can satisfy both irrotationality and two-dimensional continuity for nonstraight flow paths, when $n = 1$, $r = 1$, and $m = -p$. The main-flow streamline paths here are hyperbolic. There are, however, difficulties in interpretation of the significance of case IX flows and their associated boundary layers which will be discussed later.

Boundary Layer

Similarity parameter η. - The so-called similarity parameter defined as $\eta = y\, g(x,z)/\sqrt{\nu}$ provides information concerning the physical dimensions of the boundary layer. At the surface $\eta = 0$, and $\eta \rightarrow \infty$ as 5002 the boundary layer merges into the mainstream. However, some finite value of $\eta = \eta_0$ serves as a practical outer limit to the boundary layer (refs. 4 to 8). In terms of physical lengths, the boundary-layer thickness at any point is the height y at which the value of η reaches approximately η_0. From $y = \sqrt{\nu}\,\eta/g = \eta_0\sqrt{\nu}/g$ at the outer edge of the boundary layer, it can be seen that the boundary-layer thickness is inversely proportional to the magnitude of g at any point on a surface.

Thus, in case III, for example, for n and c positive integers, the boundary layer becomes progressively thinner with increasing x and z because $g(x,z)$ becomes progressively larger.

$g(x,z)$ constant. - When g is constant, the boundary-layer thickness does not vary over the surface. This corresponds, for example, to the stagnation-point flows (refs. 11 and 15).

Reference 15 notes that in the two-dimensional stagnation-point flow with the constant boundary-layer thickness (g constant) an exact solution for the Navier-Stokes equations is obtained where $U = ax$ and $V = ay$, for constant a. It is interesting to note that the three-dimensional stagnation-point flow of reference 10 (case IX) with constant g likewise is an exact similarity solution for the three-dimensional Navier-Stokes equations with $U = ax$, $W = cz$, $V = -y(a + c)$, $g^2 = b$, and $\eta = y\sqrt{b/\nu}$. For this case, however, the Navier-Stokes equations actually reduce to the boundary-layer equations without casting out terms because of physical considerations, such as very thin boundary layers, and so forth.

Leading edge. - The problem of relating the mathematical solutions to physical reality is intricately involved with the nature of η and $g(x,z)$. Usually the boundary layer must be considered as "beginning" somewhere in a real flow at a leading edge. At the leading edge the boundary layer has zero thickness and develops under the main flow along the surface. A line on the surface along which η takes on very large values (because $g(x,z) \rightarrow \infty$) may be considered such a leading edge. Along such a line the boundary layer has zero thickness being inversely proportional to $g(x,z)$. Because $\lim_{\eta\to\infty} F' = \lim_{\eta\to\infty} G' = 1$, then $u = UF' = U$

and $w = WG' = W$ and the boundary-layer velocity components merge smoothly into the corresponding mainstream components. An example occurs in case IV with $n = 0$ along the line $x = 0$ on the surface. This, of course, is the case treated in reference 8 as well as elsewhere.

There are situations where the mainstream velocity components are zero along a line or along an entire plane normal to the surface. It is readily seen that the boundary-layer velocity components are also zero matching the mainstream. If cases of g constant are not included for the moment, inspection of the chart reveals that either $g(x,z)$ likewise becomes zero (case IV, $n>1$ along the line $x = 0$) and so the "boundary layer" is infinitely thick along such a line or $g(x,z)$ becomes very large and the boundary layer begins at the line (case VI, $n = 1/2$ along the line $ax + cz = 0$). The situation where $g(x,z) = 0$ along a line is typical of this type of accelerated flows with a boundary layer becoming progressively thinner along the surface. In such cases, the simplest thing to do is to confine the discussion of the flows to such regions where the requirement of thin boundary layers is satisfied. The correspondence in these cases between the mathematical solutions in the thin-boundary-layer region and physical reality can be properly determined only by experiment. This, however, is equally true for flows with satisfactorily defined leading edges.

When g is constant as in the stagnation-point flows discussed previously, again no leading edge providing a "beginning" place for the boundary layer can be defined. Likewise, the physical realism of a solution is questionable when the "leading edge" is at x or $z = \infty$ or when the velocities at an otherwise well-defined leading edge ($\eta = \infty$) take on infinite values. Case IV when $n = 0$, $U = U_0$, and $\eta \to \infty$ as $x \to 0$ presents a typical case of flow over a surface with a leading edge. When $r \,\bar{<}\, 0$ however, in this case (refs. 7 and 15, e.g.) the tangential component W takes on infinite values along the leading edge.

Ordinary Differential Equations

The actual numerical solutions of the ordinary differential equations are beyond the scope of the present investigation. Numerous examples have been calculated elsewhere for particular values of the free constants in the equations. Many of these calculations are indicated in the listing "References and comments" associated with each case. While the existence of a solution and its calculation must remain to be determined individually for each case in general, certain remarks can be made here concerning particular situations.

<u>Separation of F and G.</u> - Numerical solutions will be considerably easier in those cases for which only one of the parameters, F or G, and

its derivatives, appear in one of the two equations. The listing "Separation of F and G" enumerates conditions where this will occur. Under these conditions the ordinary differential equations reduce to the forms of cases I and V.

In case I, equation (6) is a Falkner-Skan equation (ref. 13) as is equation (7) in case V. In all cases, for choices of the constants that effect separation of F and G in one of the equations (conditions which are most likely to be useful for purposes of computation) the numerical solutions for the equations in F alone or G alone are already known (refs. 14 and 15). Likewise the numerical solutions for equation (6) (cases IV and VI) and equation (7) (case II) are known.

Although it is not apparent in the table, if $a = 0$ in cases I, III, and VII, or $p = 0$ in the remaining cases, then $w = 0$ and equation (1b) and, hence, equation (7) disappear. Under these conditions, except in cases II and V (which are simply rotations of cases I and IV, respectively), equation (6) becomes the Falkner-Skan two-dimensional flow equation for which the complete solution is known (ref. 13). Similarly, whenever $m = 0$, $u = 0$, equations (1a) and (6) disappear, and equation (7) becomes the Falkner-Skan equation except in cases I and IV.

Boundary conditions not achievable. - Should the ordinary differential equations reduce to the form $F''' = 0$ (or $G''' = 0$) in any case, the solution is

$$F = k_1\eta^2 + k_2\eta + k_3$$

However, the boundary condition $\lim_{\eta \to \infty} F'(\eta) = 1$ cannot be achieved. The occurrence of such forms of the equations is listed as "Boundary conditions not achievable." The requirement that F' and G' approach their limiting values asymptotically as $\eta \to \infty$ restricts somewhat the choices of the constants as well (ref. 19). For example, when F' is to approach its limit 1 from below, the relations must be such that the curvature of the F' curve, which depends upon F''', must be negative. That is, from equation (6)

$$F''' = F'''(F,F',F'',G,G') < 0 \tag{10}$$

The practical analysis is somewhat simplified for conditions when separation of the variables occurs. Then, F' is the Falkner-Skan function. F, F', and F'' are positive, and checks can be made readily during the numerical calculation procedures for the appropriateness of the sign of F'''.

Similar statements can be made for the restrictions on the constants required for correct values of G''' and for conditions when the functions approach the limit 1 from above.

Linearity in u or w. - Under conditions for which there is linearity of equations (1) in u or w, that is, $\partial u/\partial x \equiv 0$ or $\partial w/\partial z \equiv 0$, an extension of the solution beyond strict similarity of the velocity component with respect to the corresponding x or z coordinate can be made. Reference 8 under case VII makes use of such linearity with respect to w to obtain solutions for quite general streamline shapes. In order to simplify the computations, $U = U_0$ (the case when $n = 0$) was chosen in reference 8, so that F becomes the already well-tabulated Blasius parameter F, (ref. 15) leaving only the parameter G to be computed.

CONCLUDING REMARKS

Solutions for mainstream U and W flow components for all possible boundary-layer flows having classical similarity (having affine velocity profiles) with respect to rectangular-coordinate systems are obtained in an orderly fashion. These solutions, which apply to a wide range of flows, are summarized in table I. Careful attention should be paid to certain practical considerations.

The final solutions for the boundary-layer velocity components have not been carried out. The actual numerical solutions for the ordinary differential equations derived here are considered beyond the scope of this investigation. They belong more properly to a program involving the use of high-speed computing machinery.

It is important to note that there is really no case described for such simple configurations as flows through a row of stator blades with appreciable turning, irrotationality, and very near two-dimensionality of the mainstream. The cases for flows with a satisfactorily defined leading edge include but a portion of the totality of flows described. Nevertheless, physically, the beginnings of a boundary-layer development would reasonably seem to be most important. Except for that described in reference 8, there is little experimental verification available for any three-dimensional similarity solutions and none for most solutions indicated in table I.

On the positive side, there are obviously conditions for which the solutions presented enable reasonable approximations to be calculated for three-dimensional laminar-boundary-layer behavior (see ref. 8). More important, however, if no case can be found in table I to serve as a reasonable approximation for a particular flow problem, then in all likelihood some approach other than the similarity solutions in rectangular coordinates must be sought.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, June 6, 1956

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APPENDIX A

SYMBOLS

a,b,c,d	constants
f	arbitrary function
$F, F(\eta), \tilde{F}$	function of similarity parameter, $u \equiv UF'(\eta)$
$G, G(\eta), \tilde{G}$	function of similarity parameter, $w \equiv WG'(\eta)$
$g, g(x,z)$	function of coordinates, x and z
$h,k,m,n,$ p,r	constants
U,V,W $\tilde{U},\tilde{V},\tilde{W}$	mainstream velocity components in x,y,z directions, respectively
u,v,w $\tilde{u},\tilde{v},\tilde{w}$	boundary-layer velocity components in x,y,z directions, respectively
x,y,z $\tilde{x},\tilde{y},\tilde{z}$	rectangular coordinates
α	constant
η	similarity (space) variable, $\eta \equiv \frac{y}{\sqrt{\nu}} g(x,z)$
η_0	value of η which defines practical outer limit of boundary layer
η^*	similarity variable, $\eta^* \equiv \eta^*(x,y,z)$
ν	coefficient of kinematic viscosity

Subscripts:

0	constant
$i=1,2,$ $3, \ldots$	index numbers

Superscripts:

Primes denote differentiation

APPENDIX B

SEARCH FOR SOLUTIONS

Approach to the Problem

One purpose of the present investigation was to develop a logical systematic method for obtaining more general solutions for three-dimensional incompressible thin-boundary-layer flows having affine dimensionless velocity profiles (so-called similarity solutions) then heretofore available. The utility of inspectional analysis as a method for obtaining such solutions to the partial differential equations of fluid flow is discussed in chapter IV of reference 9. The basis of the method is the axiom (obtained from group theory) that, if the hypotheses of a theory are invariant under a group of transformations, then so are its conclusions. Suppose a group of transformations can be found that leaves the form of a system of partial differential equations invariant. The method, called the "search for symmetric solutions", as applied to boundary-layer theory involves finding combinations of the reference coordinates that are likewise invariant under the group. Of even greater significance to the present investigation is the concept of an "inverse method" (ref. 9, p. 140), which states that, given a particular solution with certain properties, it may be possible to find additional solutions by a priori postulation of those properties. The procedures for applying these two methods to boundary-layer problems are described in detail later. It appears advisable first to establish the reasonableness of anticipating that the application of these methods can lead to solutions of the boundary-layer equations.

Reference 9 applies the "search for symmetric solutions" to the two-dimensional laminar-boundary-layer equations in rectangular coordinates and is able to derive a systematic, logical development of the Blasius solution in terms of a similarity parameter $\eta = y/\sqrt{x}$. Furthermore, it is known from the results of references 4 to 8 that solutions involving such similarity parameters have been obtained for the incompressible three-dimensional laminar-boundary-layer development on a nearly flat surface with a leading edge for certain restricted main flows. Thus it seems reasonable to attempt to apply the method of "search for symmetric solutions" and the "inverse" method in order to find three-dimensional boundary-layer solutions for more general main flows.

The procedure for this investigation was to use the "search for symmetric solutions" in an effort (1) to find some solutions for the incompressible three-dimensional laminar-boundary-layer flow over a surface in a logical, systematic manner, (2) to study the properties of these solutions, and (3), if possible, to invoke the inverse method to find other related solutions. In this fashion, it was intended to try to find solutions for much more general flows than had been obtained heretofore.

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Following reference 9, under transformations of similitude (changes of scale),

$$\left.\begin{array}{lll} x \to \alpha_1 \tilde{x}, & y \to \alpha_2 \tilde{y}, & z \to \alpha_3 \tilde{z} \\ u \to \alpha_4 \tilde{u}, & v \to \alpha_5 \tilde{v}, & w \to \alpha_6 \tilde{w} \\ U \to \alpha_7 \tilde{U}, & V \to \alpha_8 \tilde{V}, & W \to \alpha_9 \tilde{W} \end{array}\right\} \qquad \text{(B1)}$$

the system of equations (1) is invariant when, for example,

$$\left.\begin{array}{l} \alpha_1 = \alpha_3 = \alpha_2^2 = \alpha^2 \\ \alpha_5 = \alpha_8 = \dfrac{1}{\alpha_2} = \dfrac{1}{\alpha} \\ \alpha_4 = \alpha_6 = \alpha_7 = \alpha_9 = 1 \end{array}\right\} \qquad \text{(B2)}$$

where the last constants were chosen equal to 1 for convenience. The invariance of equations (1) under this transformation may be verified by direct substitution of equations (B1) and (B2) into equations (1).

As mentioned previously, the first step in the search for symmetric solutions is finding combinations η of the reference coordinates that are likewise invariant under the transformation of similitude using equations (B1) and (B2). Several such combinations suggest themselves at once: $\eta \sim y/\sqrt{x + z}$, $y\sqrt{x^n z^{-n-1}}$, $y^n/\sqrt{x^n + z^n}$, $y^{-n}\sqrt{x^n z^{n-r}}$, and others. Substitution from equations (B1) and (B2) verifies the invariance. The solutions are to be obtained by expressing dimensionless u and w as functions of η under conditions that will enable these functions to be evaluated. By comparison with reference 9, it can be shown that solutions which may be obtained in this fashion will be similarity-type solutions with more generalized space variables η than considered heretofore (refs. 5, 6, and 8).

Application of Method

Specifying the dimensionless velocity components $u/U = F'(\eta)$, $w/W = G'(\eta)$ as in equations (4a) and (4b) of the text and defining $\eta = y/\sqrt{x + z}$, for an example, it is found that the transformed boundary-layer equations reduce to ordinary differential equations in $F(\eta)$, $G(\eta)$, and their derivatives for

$$U = m$$

$$W = p$$

$$V = 0$$

This is the case of a family of straight main flows (actually a rotated Blasius flow) with streamline projections in the x-z plane described by $z = (p/m)\ x + z_0$.

Similarly, as suggested earlier, when

$$\eta = y\sqrt{x^n z^{-n-1}}$$

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the transformed equations reduce to ordinary differential equations for

$$U = m(x)^{n+1}(z)^{-n-1} \qquad m \neq 0$$

$$W = p(x)^n(z)^{-n} \qquad p \neq 0$$

Here is the family of mainstream flows with projected streamlines in the x-z plane described by $z^m = z_0 x^p$.

Thus, preliminary application of the method of "search for symmetric solutions" using transformations of similitude has led directly to obtaining two new similarity solutions. In order to invoke the inverse method and perhaps extend these results, the solutions were studied. As a result it was found convenient to postulate a priori the property that the boundary-layer equations can be transformed to equations containing functions of a generalized similarity parameter η, and that the transformed equations can be reduced to ordinary differential equations by determining specific forms $U(x,z)$, $W(x,z)$, and η. Four families of solutions are obtained upon application of the inverse method in this investigation and are listed as nine general cases in table I.

The associated problem of actually determining the numerical solutions to these families of differential equations is twofold. There is first the problem of the existence of such solutions for any particular case. As discussed in the text, values can be chosen for certain free constants in the equations such that the required boundary conditions cannot be achieved. Second, there is the fact that, for even moderate ranges of the constants that enter the equations as parameters, literally thousands of essayable integrations are represented in the table. The scope and purpose of this investigation is to present the flow equations in such form that their applicability to the boundary-layer development under some particular mainstream configuration will be readily apparent. The problems of numerical integration for that particular case can then be undertaken more appropriately at that time (as in refs. 4 to 8 and 10 to 15).

APPENDIX C

BOUNDARY CONDITIONS

The boundary-layer equations (1) have been transformed using the definitions

$$u \equiv UF'(\eta) \tag{4a}$$

$$w \equiv WG'(\eta) \tag{4b}$$

and the solution for v is given by equation (4c).

The mainstream flows have then been determined for which the transformed boundary-layer equations reduce to ordinary differential equations, involving $F(\eta)$, $G(\eta)$, and their derivatives. The boundary conditions on these functions of η must be chosen to correspond to the boundary conditions on u, v, and w in the original partial differential equations (1). Corresponding to the four boundary conditions,

$$u = w = 0 \quad \text{for} \quad y = 0$$

$$\lim_{y \to \infty} u = U$$

$$\lim_{W \to \infty} w = W$$

the following four boundary conditions on F and G result from equations (4a) and (4b)

$$F'(0) = G'(0) = 0$$

$$\lim_{\eta \to \infty} F' = \lim_{\eta \to \infty} G' = 1$$

The remaining boundary condition on equations (1), $v = 0$ for $y = 0$, requires that (see eq. (4c))

$$\left(\frac{\partial U}{\partial x} - \frac{U}{2}\frac{\partial \ln g^2}{\partial x}\right)F(0) + \left(\frac{\partial W}{\partial z} - \frac{W}{2}\frac{\partial \ln g^2}{\partial z}\right)G(0) = -f(x,z) \tag{C1}$$

Substituting equations (4) into equations (1a) and (1b) with $f(x,z)$ defined as in (C1) yields

$$\frac{\partial U}{\partial x}\left\{F'^2 - [F - F(0)]F'' - 1\right\} + \frac{W\partial \ln U}{\partial z}(G'F' - 1) - \frac{\partial W}{\partial z}F''[G - G(0)] +$$

$$U\frac{\partial \ln g^2}{\partial x}\frac{F''}{2}[F - F(0)] + \frac{W\partial \ln g^2}{\partial z}\frac{F''}{2}[G - G(0)] - g^2F''' = 0 \tag{C2}$$

$$\frac{\partial W}{\partial z}\left\{G'^2 - G''[G - G(0)] - 1\right\} + \frac{U\partial \ln W}{\partial x}(F'G' - 1) - \frac{\partial U}{\partial x}G''[F - F(0)] +$$

$$\frac{W\partial \ln g^2}{\partial z}\frac{G''}{2}[G - G(0)] + U\frac{\partial \ln g^2}{\partial x}\frac{G''}{2}[F - F(0)] - g^2G''' = 0 \tag{C3}$$

If the substitution is now made,

$$F = \tilde{F} + F(0)$$

$$G = \tilde{G} + G(0)$$

The above equations reduce to the form given in equations (6) and (7) with F and G replaced by $\tilde{F}$ and $\tilde{G}$, respectively. Provided that the coefficients of (C2) and (C3) have been made proportional, the solutions for equations (C2) and (C3) are uniquely determined by solutions of equations (6) and (7) in $\tilde{F}$ and $\tilde{G}$ with the boundary conditions $\tilde{F}(0) = \tilde{G}(0) = \tilde{F}'(0) = \tilde{G}'(0) = 0$; $\lim\limits_{\eta\to\infty} \tilde{F}' = \lim\limits_{\eta\to\infty} \tilde{G}' = 1$. Furthermore,

$$u = UF' = U\tilde{F}' \tag{C4}$$

$$w = WG' = W\tilde{G}' \tag{C5}$$

$$v \doteq -\frac{1}{g}\left\{\frac{\partial U}{\partial x}[F - F(0)] + \frac{\partial W}{\partial z}[G - G(0)]\right\} + \frac{U\partial g^{-1}}{\partial x}\left\{\eta F' - [F - F(0)]\right\} +$$

$$W\frac{\partial g^{-1}}{\partial z}\left\{\eta G' - [G - G(0)]\right\} \tag{C6}$$

$$\equiv -\frac{1}{g}\left(\frac{\partial U}{\partial x}\tilde{F} + \frac{\partial W}{\partial z}\tilde{G}\right) + \frac{U\partial g^{-1}}{\partial x}(\eta\tilde{F}' - \tilde{F}) + W\frac{\partial g^{-1}}{\partial z}(\eta\tilde{G}' - \tilde{G}) \tag{C7}$$

Here the solution for u, v, and w is uniquely determined by the solution for $\tilde{F}$ and $\tilde{G}$. Finally, $\tilde{F}$ and $\tilde{G}$ can be identified with F and G in equation (4c) and the provision $f(x,z) = 0$ in (4c) and $F(0) = G(0) = 0$.

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APPENDIX D

EXAMPLE OF DERIVATION OF EXPLICIT FORMS OF U, W, AND g^2

The cases of obtaining the ordinary differential equations for flows assuming $U = U(x,z)$ and $W = W(x,z)$ are presented here because they include all the variety of situations and complications that arise with other possible assumptions on the nature of U and W. This assumption is equivalent to the hypothesis that the first partial derivations of U and W do not vanish identically. For convenience in the analysis the seven different coefficients (see eq. (8)) are listed in the order of their appearance in equations (6) and (7):

1. $\partial U/\partial x$
2. $W \dfrac{\partial \ln U}{\partial z}$
3. $\partial W/\partial z$
4. $U \dfrac{\partial \ln g^2}{\partial x}$
5. $W \dfrac{\partial \ln g^2}{\partial z}$
6. g^2
7. $U \dfrac{\partial \ln W}{\partial x}$

As $\partial U/\partial x$ and $\partial W/\partial z$ have been assumed to be nonzero, an o.d.e. condition for 1 and 7 will be

$$b_1 \frac{\partial U}{\partial x} = U \frac{\partial \ln W}{\partial x}$$

Hence,

$$W = f_1(z)\ U^{b_1} \tag{D1}$$

Similarly, from 2 and 3

$$W = f_2(x)\ U^{b_2} \tag{D2}$$

Now, if $b_1 = b_2$, it follows from equations (D1) and (D2) that $f_1(z)$ and $f_2(x)$ are constants. Hence,

$$W = b_3 U^{b_1} \tag{D3}$$

If $b_1 \neq b_2$, then from equations (D1) and (D2) and ① and ③

$$U = f_3(x)f_4'(z) \tag{D4}$$

$$W = b_4 f_3'(x)f_4(z) \tag{D5}$$

<u>$b_1 \neq b_2$.</u> - First consider the case $b_1 \neq b_2$. Then by ① and ⑥

$$g^2 = b_5 f_3'(x)f_4'(z) \tag{D6}$$

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As $\partial W/\partial x = b_4 f_3''(x)f_4(z)$ from (D5), it follows that, since $\partial W/\partial x \neq 0$, then

$$\frac{\partial \ln g^2}{\partial x} = \frac{f_3''(x)}{f_3'(x)} \neq 0 \tag{D7}$$

Similarly, $\partial U/\partial z \neq 0$ implies that

$$\partial \ln g^2/\partial z = f_4''(z)/f_4'(z) \neq 0 \tag{D8}$$

Hence, from ① and ④,

$$\frac{f_3'}{f_3} = b_6 \frac{f_3''}{f_3'} \tag{D9}$$

or

$$f_3' = b_7 f_3^{\frac{1}{b_6}} \tag{D10}$$

If $b_6 = 1$,

$$f_3 = b_8 e^{b_7 x} \tag{D11}$$

If $b_6 \neq 1$,

$$f_3 = (b_9 x + b_{10})^{b_{11}} \tag{D12}$$

The constant b_{10} represents only a displacement of the coordinate axis and may be taken equal to zero without loss of generality.

Similarly, from 3 and 5, and equations (D5) and (D6)

$$f_4 = b_{12}e^{b_{13}z} \tag{D13}$$

or

$$f_4 = b_{14}z^{b_{15}} \tag{D14}$$

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From the various combinations of f_3 and f_4, the following forms for U and W result:

$$\left.\begin{aligned} U &= me^{nx}e^{cz}; & W &= ae^{nx}e^{cz} \\ U &= me^{nx}z^{r-1}; & W &= ae^{nx}z^{r} \\ U &= mx^{n}e^{cz}; & W &= px^{n-1}e^{cz} \\ U &= mx^{n}z^{r-1}; & W &= px^{n-1}z^{r} \end{aligned}\right\} \tag{D15}$$

$\underline{b_1 = b_2.}$ - Now consider the remaining case $b_1 = b_2$. If $b_1 = 0$, it follows from equation (D3) that $\partial W/\partial x = \partial W/\partial z = 0$, which contradicts the basic hypothesis. If $b_1 = 1$, $W = b_3U$. Then from 1 and 3,

$$\frac{\partial U}{\partial x} = c_1 \frac{\partial U}{\partial z} \tag{D16}$$

$$U = f_5(c_1x + z) \tag{D17}$$

$$W = b_3f_5(c_1x + z) \tag{D18}$$

Treating $(c_1x + z)$ as an independent variable, $f_5'' \equiv 0$ implies that

$$f_5 = c_2(c_1x + z) \tag{D19}$$

$$U = c_2(c_1x + z) \tag{D20}$$

$$W = b_3c_2(c_1x + z) \tag{D21}$$

If $f_5'' \not\equiv 0$, then from 1 and 6

$$\frac{\partial g^2}{\partial x} = \frac{\partial}{\partial x}\left(c_3 \frac{\partial U}{\partial x}\right) = c_3c_1^2f_5'' \not\equiv 0 \tag{D22}$$

Similarly, from ③ and ⑥,

$$\frac{\partial g^2}{\partial z} = c_4 \frac{\partial^2 W}{\partial z^2} = c_4 b_3 f_5'' \neq 0 \tag{D23}$$

Therefore, from ① and ④,

$$c_1 f_5' = c_5 c_1 \frac{f_5 f_5''}{f_5'} \tag{D24}$$

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and therefore,

$$f_5' = c_6 f_5^{1/c_5} \tag{D25}$$

If $c_5 = 1$, (D26)

$$f_5 = c_7 e^{c_6(c_1 x + z)} \tag{D26}$$

If $c_5 \neq 1$,

$$f_5 = [c_9(c_2 x + z) + c_{10}]^{c_{11}} \tag{D27}$$

As before, c_{10} can be taken equal to zero without loss of generality. The same result is obtained by use of ③ and ⑤.

In summary, for $b_1 = 1$ the following forms for U and W result:

$$\left.\begin{aligned} U &= me^{nx}e^{cz}; &\quad W &= ae^{nx}e^{cz} \\ U &= m(ax + z)^n; &\quad W &= p(ax + z)^n \end{aligned}\right\} \tag{D28}$$

The final case to be considered is $b_1 \neq 1$. For $b_1 \neq 1$, g^2 cannot be identically constant. If $g^2 = k$, a constant, then from ① and ⑥ U would have to be of the form $U = d_1 x + h_1(z)$. Then from ②, ⑥, and equation (D3), $b_3[d_1 x + h_1(z)]^{b-1} h_1'(z) = d_2$. But this requires that either $b_1 = 1$ or $h_1'(z) = 0$. Neither case is admissible. Furthermore neither $\partial g/\partial x$ nor $\partial g/\partial z$ can be identically zero if the hypothesis that the first partial derivatives of W and U do not vanish is not to be violated. If for example $g^2 = g^2(x)$, then from ④ and ⑥ it would follow that $U = U(x)$, and hence $\partial U/\partial z \equiv 0$.

From ① and ④ the following form for U is determined:

$$U = (g^2)^{d_3} h_2(z) \tag{D29}$$

From ③ and ⑤,

$$W = (g^2)^{d_4} h_3(x) \tag{D30}$$

However,

$$W = b_3 U^{b_1} \tag{D3}$$

Therefore,

$$(g^2)^{d_4 - b_1 d_3} = \frac{b_3 h_2{}^{b_1}(z)}{h_3(x)}$$

Now if $d_4 \neq b_1 d_3$, it follows at once that U and W can be expressed as a product of a function of x and a function of z. Hence, equation (D4) would hold, and the analysis would proceed accordingly as before.

If $d_4 = b_1 d_3$, then $h_2(z)$ and $h_3(x)$ must be constants and ①, ⑥, and equation (D29) give

$$d_3 (g^2)^{2-d_3} = \partial g^2/\partial x \tag{D31}$$

From ③, ⑥, and equation (D30),

$$d_6 (g^2)^{2-d_4} = \partial g^2/\partial z \tag{D32}$$

If $d_3 = 2$ or $d_4 = 2$, then $g^2 = g(x)$ or $g^2 = g(z)$; and the hypothesis is violated. If $d_3 = 1$, then from equation (D31),

$$\frac{\partial \ln g^2}{\partial x} = d_5 \tag{D33}$$

$$g^2 = h_4(z) e^{d_5 x} \tag{D34}$$

Similarly, if $d_4 = 1$, it follows from equation (D32) that

$$g^2 = h_5(x) e^{d_6 z} \tag{D35}$$

If $d_3 \neq 1,2$, integration of (D31) yields

$$g^2 = [(d_5 x + h_6(z)(d_3 - 1)]^{\frac{1}{d_3-1}} \tag{D36}$$

If $d_4 = 1,2$, integration of (D32) yields

$$g^2 = [(d_6 z + h_7(x))(d_4 - 1)]^{\frac{1}{d_4-1}} \tag{D37}$$

Equations (D34), (D35), (D36), (D29), and (D30) require that U and W be one of the forms already given in (D15) or (D28).

These forms thus obtained constitute all the possible solutions arising from the initial assumptions.

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TABLE I. - SIMILARITY SOLUTIONS IN STATIONARY RECTANGULAR COORDINATES

	Case I	Case II	Case III
U	me^{nx}	me^{nz}	$me^{nx}e^{cz}$
W	ae^{rx}	pe^{cz}	$ae^{nx}e^{cz}$
$\eta\left[yg(x,z)/\sqrt{\nu}\right]$	$y\left(\frac{be^{nx}}{\nu}\right)^{1/2} = y\left(\frac{bU}{\nu m}\right)^{1/2} = y\left[\frac{bW}{\nu ae^{(r-n)x}}\right]^{1/2}$	$y\left(\frac{be^{cz}}{\nu}\right)^{1/2} = y\left[\frac{bU}{\nu me^{(n-c)z}}\right]^{1/2} = y\left(\frac{bW}{\nu p}\right)^{1/2}$	$y\left(\frac{be^{nx}e^{cz}}{\nu}\right)^{1/2} = y\left(\frac{bU}{\nu m}\right)^{1/2} = y\left(\frac{bW}{a\nu}\right)^{1/2}$
Equation (6)	$mn\left[(F')^2 - \frac{FF''}{2} - 1\right] - bF''' = 0$	$-op\,\frac{GF''}{2} + np(G'F' - 1) - bF''' = 0$	$(mn+ac)\left[(F')^2 - \frac{FF''}{2} - 1\right] - bF''' = 0$
Equation (7)	$mr(F'G' - 1) - mn\,\frac{G''F}{2} - bG''' = 0$	$op\left[(G')^2 - \frac{GG''}{2} - 1\right] - bG''' = 0$	$F = G$
Projection of main stream on surface	$z = \frac{a}{m(r-n)}\,e^{(r-n)x} + z_0$	$x = \frac{m}{p(n-c)}\,e^{(n-c)z} + x_0$	$\frac{x}{m} = \frac{z}{a} + x_0$
Irrotationality	$ar = 0$	$mn = 0$	$cm = an$
Linearity in u	$mn = 0$	Linear	$mn = 0$
Linearity in w	Linear	$op = 0$	$ac = 0$
Boundary conditions not achievable (6) (7)	$mn = 0$ $mr = mn = 0$	$op = np = 0$ $op = 0$	$mn = -ac$
Separation of F and G (6) (7)	Separated $mn = mr = 0$	$op = np = 0$ Separated	Does not apply
References and comments	General solution of eq. (6) in refs. 13 and 14. If $n = r$, then $F = G$.	Case I with rotation of coordinate axis. Eq. (7) is solved in refs. 13 and 14. If $c = n$, $F = G$, and eq. (6) is completely solved in refs. 13 and 14.	This is a case of rotated two-dimensional flow solved in refs. 13 and 14 with $b = mn + ac$.

TABLE I. - Continued. SIMILARITY SOLUTIONS IN STATIONARY RECTANGULAR COORDINATES

	Case IV	Case V	Case VI
U	mx^n	mz^n	$m(ax + cz)^n$
W	px^r	pz^n	$p(ax + cz)^n$
$\eta\left[yg(x,z)/\sqrt{\nu}\right]$	$y\left(\frac{bx^{n-1}}{\nu}\right)^{1/2} = y\left(\frac{bU}{\nu m x}\right)^{1/2} = y\left(\frac{bW}{px^{r+1-n}}\right)^{1/2}$	$y\left(\frac{bz^{r-1}}{\nu}\right)^{1/2} = y\left(\frac{bU}{\nu m z^{n-r+1}}\right)^{1/2} = y\left(\frac{bW}{\nu pz}\right)^{1/2}$	$y\left[\frac{b(ax+cz)^{n-1}}{\nu}\right]^{1/2} = y\left[\frac{bU}{\nu m(ax+cz)}\right]^{1/2} = y\left[\frac{bW}{\nu p(ax+cz)}\right]^{1/2}$
Equation (6)	$mn\left[(F')^2 - 1\right] - m(n+1)\frac{FF''}{2} - bF''' = 0$	$-p(r+1)\frac{GF''}{2} + pn(G'F' - 1) - bF''' = 0$	$n(am+cp)\left[(F')^2 - 1\right] - (n+1)(am+cp)\frac{FF''}{2} - bF''' = 0$
Equation (7)	$mr(F'G' - 1) - m(n+1)\frac{G''F}{2} - bG''' = 0$	$pr\left[(G')^2 - 1\right] - p(r+1)\frac{GG''}{2} - bG''' = 0$	$F = G$
Projection of main stream on surface	$z = \frac{p}{m}\frac{x^{(r-n+1)}}{r-n+1} + z_0$	$x = \frac{p}{m}\frac{z^{(r-n+1)}}{r-n+1} + x_0$	$z = \frac{p}{m}x + z_0$
Irrotationality	$pr = 0$	$mn = 0$	$cm = pa$
Linearity in u	$m = 0$	Linear	$am = 0$
Linearity in w	Linear	$p = 0$	$cp = 0$
Boundary conditions not achievable (6) (7)	$m = 0$ $mr = m(n+1) = 0$	$p = 0$ $p = 0$	$am = -cp$
Separation of F and G (6) (7)	Separated $mr = m(n+1) = 0$	$p(r+1) = pn = 0$ Separated	Separated
References and comments	If $r = n$, $F = G$. Ref. 5: $n = 0$, $m = r = b$. Ref. 6: $n = 0$, $m = b$, $\eta = 2\xi$ (ref. 6). Ref. 7: $n = 0$, $m = b$, $r = n$ (ref. 7). Ref. 8: $n = 0$, $m = b$. Ref. 12: $m = b$, $r = 0$, $G' = g$ (ref. 12). Ref. 13: $m = b$, general solution of eq. (6). Ref. 14: $m(n+1) = 2b$. Ref. 15: plane stagnation flow: $p = 0$, $n = 1$, $b = m$. Blasius flow: $p = n = 0$, $b = m$.	This is case IV under rotation of coordinate axes.	This case is solved completely in refs. 13 and 14 where $b = (n+1)(am+cp)$. Ref. 11: stagnation point flow: $cp = 0$, $n = 1$, $am = b$. Ref. 11: axisymmetric case: $cp = 0$, $n = 1/3$, $am/b = 3/2$. Ref. 16: plane stagnation flow: $c = p = 0$, $n = 1$, $b = m$.

TABLE I. - Concluded. SIMILARITY SOLUTIONS IN STATIONARY RECTANGULAR COORDINATES

	Case VII	Case VIII	Case IX
U	$me^{nx}z^{r-1}$	$mx^{n}e^{cz}$	$mx^{n}z^{r-1}$
W	$ae^{nx}z^{r}$	$px^{n-1}e^{cz}$	$px^{n-1}z^{r}$
$\eta\left[yg(x,z)/\sqrt{\nu}\right]$	$y\left(\frac{be^{nx}z^{r-1}}{\nu}\right)^{1/2} = y\left(\frac{bU}{m\nu}\right)^{1/2} = y\left(\frac{bW}{a\nu z}\right)^{1/2}$	$y\left(\frac{bx^{n-1}e^{cz}}{\nu}\right)^{1/2} = y\left(\frac{bU}{m\nu}\right)^{1/2} = y\left(\frac{bW}{\nu p}\right)^{1/2}$	$y\left(\frac{bx^{n-1}z^{r-1}}{\nu}\right)^{1/2} = y\left(\frac{bU}{\nu mx}\right)^{1/2} = y\left(\frac{bW}{\nu pz}\right)^{1/2}$
Equation (6)	$mn\left[(F')^2 - \frac{FF''}{2} - 1\right] + a(r-1)(G'F'-1) - a(r+1)\frac{GF''}{2} - bF''' = 0$	$mn\left[(F')^2 - 1\right] - m(n+1)\frac{FF''}{2} + cp\left(G'F' - \frac{GF''}{2} - 1\right) - bF''' = 0$	$mn\left[(F')^2 - 1\right] - m(n+1)\frac{FF''}{2} +$ $p(r-1)(G'F'-1) - p(r+1)\frac{GF''}{2} - bF''' = 0$
Equation (7)	$ar\left[(G')^2 - 1\right] - a(r+1)\frac{GG''}{2} + mn\left(F'G' - \frac{FG''}{2} - 1\right) - bG''' = 0$	$cp\left[(G')^2 - \frac{GG''}{2} - 1\right] + m(n-1)(F'G'-1) - m(n+1)\frac{FG''}{2} - bG''' = 0$	$pr\left[(G')^2 - 1\right] - p(r+1)\frac{GG''}{2} +$ $m(n-1)(F'G'-1) - m(n+1)\frac{G''F}{2} - bG''' = 0$
Projection of main stream on surface	$z = z_0e^{ax/m}$; $\left(\frac{ax}{m} = \ln z + z_0\right)$	$x = x_0e^{mz/p}$; $\left(\frac{mz}{p} = \ln x + x_0\right)$	$x_0z^{m} = x^{p}$
Irrotationality	$m(r-1) = an = 0$	$mc = p(n-1) = 0$	$m(r-1) = p(n-1) = 0$
Linearity in u	$mn = 0$	$n = 0$	$m = 0$
Linearity in w	$a = 0$	$cp = 0$	$p = 0$
Boundary conditions not achievable (6) (7)	$mn = a = 0$ $mn = a = 0$	$m = cp = 0$ $m = cp = 0$	$m = p = 0$ $m = p = 0$
Separation of F and G (6) (7)	$a = 0$ $mn = 0$	$cp = 0$ $m = 0$	$p = 0$ $m = 0$
References and comments	If $m = 0$, then eqs. (1a) and (6) disappear, $F \equiv 0$, and eq. (7) is completely solved in refs. 13 and 14. If $a = 0$, then eqs. (1b) and (7) disappear, $G \equiv 0$, and eq. (6) is solved in refs. 13 and 14.	See case VII. Ref. 15: plane stagnation flow: $c = 0$, $n = 1$, $m = b$.	If $p = 0$, eqs. (1b) and (7) disappear, $G \equiv 0$, and eq. (6) is solved in refs. 13 and 14. If $m = 0$, eqs. (1a) and (6) disappear, $F \equiv 0$, and eq. (7) is solved in refs. 13 and 14. If $p = m$, $F = b$, ref. 13. Ref. 10: $r = n$, $m = b$, $p = c$ (ref. 10). Ref. 11: plane stagnation point flow: $p = 0$, $n = 1$, $m = b$, in eq. (6). Ref. 15, p. 71: plane stagnation flow: $n = 1$, $b = m$, $p = 0$.